Complex Numbers & Roots of Unity

Olympiad AIME | Workshop Handout

Olympiad Edge · Advanced Competition Series · 2025–2026

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1 The Complex Plane Toolkit

Every complex number $z = a + bi$ can be written as

$$z = r(\cos\theta + i\sin\theta) = r\,e^{i\theta} = r\,\mathrm{cis}\,\theta,$$

where $r = |z| = \sqrt{a^2+b^2}$ and $\theta = \arg(z)$.

Key operations in polar form:

- **Multiply:** $r_1 e^{i\theta_1} \cdot r_2 e^{i\theta_2} = r_1 r_2\, e^{i(\theta_1+\theta_2)}$ (multiply moduli, add arguments)
- **Divide:** $\dfrac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}} = \dfrac{r_1}{r_2}\, e^{i(\theta_1-\theta_2)}$
- **Power:** $\left(re^{i\theta}\right)^n = r^n e^{in\theta}$ (De Moivre's Theorem)

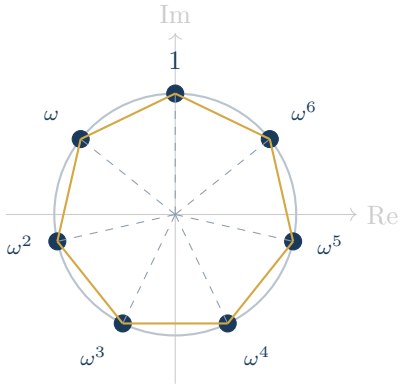
Multiplying by $e^{i\theta}$ **rotates** a point by angle θ around the origin. If a problem mentions "rotating a point," "equilateral triangle on the plane," or "regular polygon vertices," your brain should immediately fire: **COMPLEX NUMBERS.**

The nth roots of unity are the n solutions to $z^n = 1$:

$$\omega_k = e^{2\pi ik/n} = \cos\left(\frac{2\pi k}{n}\right) + i\sin\left(\frac{2\pi k}{n}\right), \qquad k = 0, 1, \ldots, n-1.$$

We write $\omega = e^{2\pi i/n}$ for the **primitive** nth root. Then the full set is $\{1, \omega, \omega^2, \ldots, \omega^{n-1}\}$.

Geometric picture: these are the vertices of a regular n-gon inscribed in the unit circle.



7th roots of unity (regular heptagon)

2 The Five Power Tools

These are the identities that show up *over and over* at the AMC 12/AIME level.

$$1 + \omega + \omega^2 + \cdots + \omega^{n-1} = 0.$$

Why? This is a geometric series with ratio $\omega \neq 1$: $\dfrac{\omega^n - 1}{\omega - 1} = \dfrac{0}{\omega - 1} = 0$.
Consequence: The sum of the vertices of any regular n-gon centered at the origin is $\mathbf{0}$.

$$z^n - 1 = (z-1)(z-\omega)(z-\omega^2)\cdots(z-\omega^{n-1}) = \prod_{k=0}^{n-1}(z-\omega^k).$$

Plugging in specific values of z gives you product identities **for free**. This is the single most powerful trick in this handout.
Key substitutions:

- Dividing by $(z-1)$ gives the **cyclotomic shortcut**: $z^{n-1} + z^{n-2} + \cdots + z + 1 = \prod_{k=1}^{n-1}(z-\omega^k)$.
- $z = a$: $\prod_{k=0}^{n-1}(a-\omega^k) = a^n - 1$. *(Evaluate any product over all roots instantly!)*
- $z = 1$ (after canceling): $\prod_{k=1}^{n-1}(1-\omega^k) = n$.

For any integer m:

$$\sum_{k=0}^{n-1} \omega^{mk} = \begin{cases} n & \text{if } n \mid m, \\ 0 & \text{if } n \nmid m. \end{cases}$$

Translation: This lets you "filter out" every nth term of a sum or generating function. If a problem asks you to sum every kth coefficient of a polynomial, *roots filter*.

Extraction formula: $\displaystyle\sum_{\substack{j\geq 0 \\ n\mid j}} a_j = \frac{1}{n}\sum_{t=0}^{n-1} f(\omega^t)$, where $f(x) = \sum a_j x^j$.

For any nth root of unity ω^k on the unit circle:

$$|1-\omega^k| = \left|1 - e^{2\pi i k/n}\right| = 2\sin\left(\frac{k\pi}{n}\right).$$

This bridges the **algebraic world** (products of $(1-\omega^k)$) and the **trigonometric world** (products of sines).

Killer combo with Tool 2: $\displaystyle\prod_{k=1}^{n-1} 2\sin\frac{k\pi}{n} = n.$

Since $\overline{\omega^k} = \omega^{n-k}$, the roots come in conjugate pairs (except 1, and -1 if n is even). This means:

- Products over all roots are always **real**.
- $|1-\omega^k|^2 = (1-\omega^k)(1-\omega^{n-k}) = 2 - 2\cos\left(\frac{2\pi k}{n}\right)$.
- When computing products of roots with **positive imaginary part**, select $k = 1, \ldots, \lfloor (n-1)/2 \rfloor$.

If a problem says "find the product of all distances from one vertex of a regular n-gon to all other vertices," you're computing $\prod_{k=1}^{n-1} |1-\omega^k| = n$. **One-line solution.**

Any time a problem asks for the sum of every kth coefficient of a polynomial $f(x)$, evaluate f at the kth roots of unity and average: $\frac{1}{k}\sum_{t=0}^{k-1} f(\omega^t)$.

Decision Flowchart

You see...	Reach for...
Regular polygon, distances, products	Roots of unity + Tool 2
"Sum of every kth coefficient"	Roots filter (Tool 3)
Rotate a point / equilateral triangle	Multiply by $e^{i\theta}$
$\prod_k(\text{something} - \omega^k)$	Plug into $z^n - 1$ (Tool 2)
$\cos(2\pi/n)$ or $\sin(2\pi/n)$ expression	Tool 4 (Distance–Sine Bridge)
Product of trig functions	Tool 4 + Tool 2 combo
Roots with positive imaginary part	Tool 5 (Conjugate Pairing)

3 Worked Examples

We solve Problems 1 and 2 in full to demonstrate the workflow before you tackle the remaining five.

Worked Example 1: Complex Rotation — 1994 AIME #8

Problem. The points $(0,0)$, $(a,11)$, and $(b,37)$ are the vertices of an equilateral triangle. Find the value of ab.

Solution.
Step 1 — Model. Represent the vertices as complex numbers: $P_1 = 0$, $P_2 = a+11i$, $P_3 = b+37i$.
Step 2 — Recognize. One vertex is at the origin, so P_3 is obtained from P_2 by a $\pm 60^\circ$ rotation about the origin. This is **Recognition Trigger #1**: multiply by $e^{i\pi/3}$!

$$P_3 = P_2 \cdot e^{\pm i\pi/3} = (a+11i)\left(\tfrac{1}{2} \pm \tfrac{\sqrt{3}}{2}\, i\right).$$

Step 3 — Compute (taking +). Expanding:

$$\begin{aligned} b+37i &= (a+11i)\left(\tfrac{1}{2} + \tfrac{\sqrt{3}}{2} i\right) \\ &= \frac{a}{2} + \frac{a\sqrt{3}}{2}\, i + \frac{11}{2}\, i + \frac{11\sqrt{3}}{2}\, i^2 \\ &= \left(\frac{a}{2} - \frac{11\sqrt{3}}{2}\right) + \left(\frac{a\sqrt{3}}{2} + \frac{11}{2}\right) i. \end{aligned}$$

Equating imaginary parts: $37 = \frac{a\sqrt{3}}{2} + \frac{11}{2}$, so $a\sqrt{3} = 63$, giving $a = 21\sqrt{3}$.
Equating real parts: $b = \frac{21\sqrt{3}}{2} - \frac{11\sqrt{3}}{2} = 5\sqrt{3}$.
Step 4 — Answer.

$$ab = (21\sqrt{3})(5\sqrt{3}) = 105 \cdot 3 = \boxed{315}.$$

Check: the − case gives $a = -21\sqrt{3}$, $b = -5\sqrt{3}$, same product.

Worked Example 2: Cyclotomic Factorization — 1996 AIME #11

Problem. Let P be the product of the roots of $z^6+z^4+z^3+z^2+1=0$ that have a positive imaginary part. Suppose $P = r(\cos\theta^\circ + i\sin\theta^\circ)$ where $0<r$ and $0\le\theta<360$. Find θ.

Solution.

Step 1 — Factor into cyclotomics. Test whether the 5th roots of unity are roots. If $\omega = e^{2\pi i/5}$, then $\omega^5=1 \Rightarrow \omega^6=\omega$, so:

$$\omega^6+\omega^4+\omega^3+\omega^2+1 = \omega+\omega^4+\omega^3+\omega^2+1 = 0 \quad \checkmark \text{ (by Tool 1).}$$

So $\Phi_5(z) = z^4+z^3+z^2+z+1$ divides our polynomial. Long division:

$$z^6+z^4+z^3+z^2+1 = \underbrace{(z^4+z^3+z^2+z+1)}_{\Phi_5(z)}\cdot\underbrace{(z^2-z+1)}_{\Phi_6(z)}.$$

Step 2 — List all roots.

- Roots of Φ_5: $\operatorname{cis}72^\circ$, $\operatorname{cis}144^\circ$, $\operatorname{cis}216^\circ$, $\operatorname{cis}288^\circ$ (primitive 5th roots).
- Roots of Φ_6: $\operatorname{cis}60^\circ$, $\operatorname{cis}300^\circ$ (primitive 6th roots).

Step 3 — Select positive imaginary part. We need $0^\circ<\theta<180^\circ$:

$$\operatorname{cis}60^\circ, \quad \operatorname{cis}72^\circ, \quad \operatorname{cis}144^\circ.$$

Step 4 — Multiply. All lie on the unit circle, so $r=1$. The product is:

$$P = \operatorname{cis}(60^\circ+72^\circ+144^\circ) = \operatorname{cis}\,276^\circ, \quad \theta = \boxed{276}.$$

Common Trap

Don't confuse ω^k with $k\omega$! When computing ω^6 for a 5th root, reduce: $\omega^6=\omega^{5+1}=\omega$. **Always reduce exponents mod n** before combining.

4 Practice Problem Set

All problems are from official competitions. Roughly ordered by difficulty. Target: 5–8 minutes each.

1990 AIME, Problem 12

A regular 12-gon is inscribed in a circle of radius 12. The sum of the lengths of all sides and diagonals of the 12-gon can be written in the form

$$a+b\sqrt{2}+c\sqrt{3}+d\sqrt{6},$$

where a, b, c, and d are positive integers. Find $a+b+c+d$.

Hint: Place vertices at $12\omega^k$. Use $|1-\omega^m| = 2\sin(m\pi/n)$ and count pairs at each separation.

2018 AIME I, Problem 12

For every subset T of $U = \{1, 2, 3, \ldots, 18\}$, let $s(T)$ be the sum of the elements of T, with $s(\emptyset) = 0$. If T is chosen at random among all subsets of U, the probability that $s(T)$ is divisible by 3 is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find m.

Hint: The generating function is $f(x) = \prod_{j=1}^{18}(1 + x^j)$. Apply the roots filter with $\omega = e^{2\pi i/3}$.

2009 AIME II, Problem 13

Let A and B be the endpoints of a semicircular arc of radius 2. The arc is divided into seven congruent arcs by six equally spaced points $C_1, C_2, \ldots, C_6$. All chords of the form $\overline{AC_k}$ or $\overline{BC_k}$ are drawn. Let n be the product of the lengths of these twelve chords. Find the remainder when n is divided by 1000.

Hint: Use $|AC_k| = 4\sin(k\pi/14)$ and $|BC_k| = 4\cos(k\pi/14)$. The double-angle formula merges these into $\sin(k\pi/7)$.

2015 AIME I, Problem 13

With all angles measured in degrees, the product $\displaystyle\prod_{k=1}^{45} \csc^2(2k-1)^\circ = m^n$, where m and n are integers greater than 1. Find $m + n$.

Hint: This covers $\sin 1^\circ, \sin 3^\circ, \ldots, \sin 89^\circ$. Connect to the roots of $z^{90} + 1$ and use Tool 4.

2024 AIME II, Problem 13

Let $\omega \neq 1$ be a 13th root of unity. Find the remainder when

$$\prod_{k=0}^{12}(2 - 2\omega^k + \omega^{2k})$$

is divided by 1000.

Hint: Factor $z^2 - 2z + 2 = (z - (1 + i))(z - (1 - i))$, then use $\prod_{k=0}^{n-1}(a - \omega^k) = a^n - 1$.

5 Solutions

Solution to Problem 1 (1990 AIME #12) — Answer: **720**

Place the vertices at $z_k = 12\,e^{2\pi ik/12}$ for $k = 0, 1, \ldots, 11$. The distance between two vertices separated by m positions around the 12-gon is:

$$d_m = |12(1-\omega^m)| = 12 \cdot 2\sin\frac{m\pi}{12} = 24\sin\frac{m\pi}{12},$$

where $\omega = e^{2\pi i/12}$. There are 12 pairs at each separation $m = 1, 2, 3, 4, 5$ and 6 pairs at $m = 6$ (diametrically opposite).
Computing each distance using exact sine values:

m	$24\sin(m\pi/12)$	Pairs	Contribution
1	$24\sin 15^\circ = 6(\sqrt{6}-\sqrt{2})$	12	$72\sqrt{6}-72\sqrt{2}$
2	$24\sin 30^\circ = 12$	12	144
3	$24\sin 45^\circ = 12\sqrt{2}$	12	$144\sqrt{2}$
4	$24\sin 60^\circ = 12\sqrt{3}$	12	$144\sqrt{3}$
5	$24\sin 75^\circ = 6(\sqrt{6}+\sqrt{2})$	12	$72\sqrt{6}+72\sqrt{2}$
6	$24\sin 90^\circ = 24$	6	144

Summing all contributions:

$$\text{Total} = (144+144) + (-72+144+72)\sqrt{2} + 144\sqrt{3} + (72+72)\sqrt{6} = 288 + 144\sqrt{2} + 144\sqrt{3} + 144\sqrt{6}.$$

Therefore $a+b+c+d = 288+144+144+144 = \boxed{720}$.

Solution to Problem 2 (2018 AIME I #12) — Answer: **683**

The generating function encoding all subset sums is $f(x) = \prod_{j=1}^{18}(1+x^j)$.
By the **roots of unity filter** with $\omega = e^{2\pi i/3}$, the number of subsets with $s(T) \equiv 0 \pmod 3$ is:

$$N_0 = \frac{1}{3}\big[f(1) + f(\omega) + f(\omega^2)\big].$$

$f(1)$**:** Every factor is $1+1=2$, so $f(1) = 2^{18}$.
$f(\omega)$**:** Since $\omega^3 = 1$, each factor $(1+\omega^j)$ depends only on $j \bmod 3$. Among $\{1, \ldots, 18\}$, there are exactly 6 values in each residue class. Using $1+\omega+\omega^2 = 0$:

$$j \equiv 0\colon\ 1+1=2, \qquad j \equiv 1\colon\ 1+\omega = -\omega^2, \qquad j \equiv 2\colon\ 1+\omega^2 = -\omega.$$

$$f(\omega) = 2^6 \cdot (-\omega^2)^6 \cdot (-\omega)^6 = 64 \cdot \omega^{12} \cdot \omega^6 = 64 \cdot 1 \cdot 1 = 64.$$

By conjugate symmetry, $f(\omega^2) = 64$.

$$N_0 = \frac{2^{18}+64+64}{3} = \frac{262272}{3} = 87424.$$

Probability: $\frac{87424}{2^{18}} = \frac{87424}{262144}$. Dividing both by 128: $\frac{683}{2048}$.
Since 683 is prime and $2048 = 2^{11}$, this is in lowest terms. $m = \boxed{683}$.

Solution to Problem 3 (2009 AIME II #13) — Answer: **672**

Place the semicircle on a circle of radius 2 centered at the origin, with $A = 2$ and $B = -2$. The six intermediate points are $C_k = 2e^{ik\pi/7}$ for $k = 1, \dots, 6$.
Using $|1 - e^{i\theta}| = 2\sin(\theta/2)$:

$$|AC_k| = |2 - 2e^{ik\pi/7}| = 4\sin\frac{k\pi}{14}, \qquad |BC_k| = |-2 - 2e^{ik\pi/7}| = 4\cos\frac{k\pi}{14}.$$

Each pair contributes:

$$|AC_k| \cdot |BC_k| = 16\sin\frac{k\pi}{14}\cos\frac{k\pi}{14} = 8\sin\frac{k\pi}{7}.$$

So the total product is:

$$n = \prod_{k=1}^{6} 8\sin\frac{k\pi}{7} = 8^6 \cdot \prod_{k=1}^{6}\sin\frac{k\pi}{7}.$$

By **Tool 4 + Tool 2**: $\displaystyle\prod_{k=1}^{6} 2\sin\frac{k\pi}{7} = 7$, so $\displaystyle\prod_{k=1}^{6}\sin\frac{k\pi}{7} = \frac{7}{2^6} = \frac{7}{64}$.

$$n = 8^6 \cdot \frac{7}{64} = \frac{262144 \cdot 7}{64} = 4096 \cdot 7 = 28672.$$

$28672 \bmod 1000 = \boxed{672}$.

Solution to Problem 4 (2015 AIME I #13) — Answer: **91**

We need $\prod_{k=1}^{45}\sin^2(2k-1)^\circ$, which covers the 45 values $\sin 1^\circ, \sin 3^\circ, \dots, \sin 89^\circ$.
The polynomial $z^{90} + 1$ has roots $e^{i(2j-1)\pi/90}$ for $j = 1, \dots, 90$. Evaluating at $z = 1$:

$$2 = \prod_{j=1}^{90}\left(1 - e^{i(2j-1)\pi/90}\right).$$

The 90 roots come in 45 conjugate pairs (angles α and $2\pi - \alpha$). Grouping:

$$2 = \prod_{j=1}^{45}|1 - e^{i(2j-1)\pi/90}|^2 = \prod_{j=1}^{45} 4\sin^2\frac{(2j-1)\pi}{180} = 4^{45}\prod_{j=1}^{45}\sin^2(2j-1)^\circ.$$

Therefore:

$$\prod_{j=1}^{45}\sin^2(2j-1)^\circ = \frac{2}{4^{45}} = \frac{2}{2^{90}} = \frac{1}{2^{89}}.$$

Taking reciprocals: $\displaystyle\prod_{k=1}^{45}\csc^2(2k-1)^\circ = 2^{89}$.
So $m = 2$, $n = 89$, and $m + n = \boxed{91}$.

Solution to Problem 5 (2024 AIME II #13) — Answer: **321**

Since ω is a 13th root of unity, $\{\omega^0, \omega^1, \ldots, \omega^{12}\}$ is the full set of 13th roots. Setting $z = \omega^k$, factor the quadratic $z^2 - 2z + 2$:

$$z^2 - 2z + 2 = (z - (1+i))(z - (1-i)).$$

The product splits:

$$\prod_{k=0}^{12}(\omega^{2k} - 2\omega^k + 2) = \underbrace{\prod_{k=0}^{12}(\omega^k - (1+i))}_{A} \cdot \underbrace{\prod_{k=0}^{12}(\omega^k - (1-i))}_{B}.$$

By **Tool 2** (the master factorization), $\prod_{k=0}^{n-1}(X - \omega^k) = X^n - 1$. Reversing signs for odd $n = 13$:

$$\prod_{k=0}^{12}(\omega^k - a) = (-1)^{13}\prod_{k=0}^{12}(a - \omega^k) = -(a^{13} - 1) = 1 - a^{13}.$$

Computing $(1+i)^{13}$:

$$\begin{aligned}(1+i)^2 &= 2i, \quad (1+i)^4 = -4, \quad (1+i)^8 = 16,\\ (1+i)^{12} &= (1+i)^8 \cdot (1+i)^4 = 16 \cdot (-4) = -64,\\ (1+i)^{13} &= -64(1+i) = -64 - 64i.\end{aligned}$$

So $A = 1 - (-64 - 64i) = 65 + 64i$.
By conjugation, $(1-i)^{13} = -64 + 64i$, so $B = 1 - (-64 + 64i) = 65 - 64i$.

$$A \cdot B = (65 + 64i)(65 - 64i) = 65^2 + 64^2 = 4225 + 4096 = 8321.$$

$8321 \bmod 1000 = \boxed{321}$.

Summary: Three Moves That Solve Everything

The Three-Move Playbook

1. **The Substitution Move.** $\prod_{k=0}^{n-1}(a-\omega^k) = a^n - 1$.
   Plugging a value into $z^n - 1$ instantly evaluates products that look intractable.
   Used in: Problems 1, 3, 4, 5, and both Worked Examples.

2. **The Distance–Sine Bridge.** $|1 - e^{i\theta}| = 2\sin(\theta/2)$.
   Converts algebraic products over roots of unity into trigonometric products.
   Used in: Problems 1, 3, 4.

3. **The Roots Filter.** $\frac{1}{n}\sum_{t=0}^{n-1} f(\omega^t) = \sum_{n|k} a_k$.
   Extracts every nth coefficient of a generating function.
   Used in: Problem 2.

Master these three and you can solve the vast majority of roots-of-unity problems from AMC 12 through AIME.